Teacher notes
Topic E

What about momentum conservation in an atomic transition?

Consider the transition from n = 2 to n = 1 in hydrogen. The energy released is 10.2 eV and is shared by the photon and the recoiling hydrogen atom. How much energy does the recoiling atom take? How good is the approximation of neglecting the atom recoil energy?

It is a standard classroom (and exam) exercise to calculate the wavelength of the emitted photon. We all write down $E=\frac{hc}{\lambda}\Rightarrow\lambda=\frac{hc}{E}=\frac{1.24\times10^{-6}}{10.2}=1.22\times10^{-7}\text{ m}$ but this assumes that all the energy available (10.2 eV) goes into the photon energy. However, the total momentum of the atom was zero before the emission of the photon so the total momentum of the atom and the photon after the emission must also be zero. The hydrogen atom will recoil in the opposite direction to that of the photon. So, since the atom will take kinetic energy the photon's energy will be less than 10.2 eV. Thus, the calculation for the wavelength above is only approximate. How good is this approximation? What kinetic energy K_{H} does the atom take? We must have energy and momentum conservation.

We will use $E=10.2\text{ eV}$ and $M_{\text{H}}c^2=939\text{ MeV}$. ($M_{\text{H}}$ is the mass of the hydrogen atom.)

We can answer this in two ways. We expect that the recoil energy will be small so from energy conservation $E=E_\gamma+K_{\text{H}}$ we deduce $E\approx E_\gamma$ if $K_{\text{H}}\approx0$. So, let us make this assumption and see if it is consistent. The photon momentum is $p=\frac{E_\gamma}{c}\approx\frac{E}{c}$ and this is also the magnitude of the atom's momentum. Then the recoil energy of the atom is

$$\frac{p^2}{2M_{\text{H}}}=\frac{1}{2M_{\text{H}}}(\frac{E}{c})^2=\frac{E^2}{2M_{\text{H}}c^2}\approx\frac{1}{2}\times\frac{10.2^2}{939\times10^6}\approx5.5\times10^{-8}\text{ eV}.$$

This is consistent with our assumption that $K_{\text{H}}\approx0$.

For a more formal answer we use energy and momentum conservation.

Energy conservation: $E=E_\gamma+K_{\text{H}}$ (1)

where $K_{\mathrm{H}} = \dfrac{p^2}{2M_{\mathrm{H}}} \Rightarrow p = \sqrt{2M_{\mathrm{H}}K_{\mathrm{H}}}$ (M_{H} is the mass of the hydrogen atom and K_{H} its kinetic energy).

And

Momentum conservation: $\dfrac{E_\gamma}{c} = \sqrt{2K_{\mathrm{H}}M_{\mathrm{H}}}$ hence $E_\gamma = c\sqrt{2K_{\mathrm{H}}M_{\mathrm{H}}}$ (2)

So (2) in (1)

$$E = c\sqrt{2K_{\mathrm{H}}M_{\mathrm{H}}} + K_{\mathrm{H}}$$

This gives

$$(E - K_{\mathrm{H}})^2 = 2K_{\mathrm{H}}M_{\mathrm{H}}c^2$$

Or

$$E^2 - 2EK_{\mathrm{H}} + E_{\mathrm{H}}^2 = 2K_{\mathrm{H}}M_{\mathrm{H}}c^2$$
$$K_{\mathrm{H}}^2 + 2K_{\mathrm{H}}(M_{\mathrm{H}}c^2 - E) + E^2 = 0$$

We can estimate K_{H} by actually solving $K_{\mathrm{H}}^2 + 2K_{\mathrm{H}}(M_{\mathrm{H}}c^2 - E) + E^2 = 0$:

The result is

$$K_{\mathrm{H}} = M_{\mathrm{H}}c^2 - E \pm \sqrt{(M_{\mathrm{H}}c^2 - E)^2 - E^2}$$
$$K_{\mathrm{H}} = M_{\mathrm{H}}c^2 - E \pm \sqrt{(M_{\mathrm{H}}c^2)^2 - 2EM_{\mathrm{H}}c^2}$$
$$K_{\mathrm{H}} = M_{\mathrm{H}}c^2 - E \pm M_{\mathrm{H}}c^2\sqrt{1 - \frac{2E}{M_{\mathrm{H}}c^2}} \qquad \text{(expand the square root)}$$
$$K_{\mathrm{H}} \approx M_{\mathrm{H}}c^2 - E \pm M_{\mathrm{H}}c^2\left(1 - \frac{E}{M_{\mathrm{H}}c^2} - \frac{1}{8}\left(\frac{2E^2}{M_{\mathrm{H}}c^2}\right)^2 + \ldots\right)$$
$$K_{\mathrm{H}} \approx \frac{1}{2}\frac{E^2}{M_{\mathrm{H}}c^2} \approx \frac{1}{2} \times \frac{10.2^2}{939 \times 10^6} \approx 5.5 \times 10^{-8}\ \text{eV}$$

This is exactly the answer we got earlier!

So, the hydrogen atom recoils with a negligible energy, momentum is conserved and the naïve calculation for the photon wavelength is correct to an excellent approximation

($E_\gamma = \dfrac{hc}{\lambda} \Rightarrow \lambda = \dfrac{hc}{E_\gamma} = \dfrac{1.24 \times 10^{-6}}{10.2 - 5.5 \times 10^{-8}} = 1.22 \times 10^{-7}$ m),

and all is good.